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LMI-based criteria for synchronization of complex dynamical networks

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Abstract

This paper considers the problem of controlling a complex dynamical network by means of pinning. We point out that the synchronization criterion given in the form of matrix eigenvalues can be equivalent to a linear matrix inequality criterion. We further investigate network synchronization via pinning and prove several linear matrix inequality theorems. In particular, we theoretically provide two typical pinning strategies based on whether the graph which is made up of unpinned nodes and edges between them is irreducible or not. Numerical simulations including *k*-regular networks, star-shaped networks and BA scale-free networks, are shown for illustration and verification.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The control over dynamics which take place in complex systems consisting of large ensembles of interacting units is one of the most important issues in various fields of applied science and engineering. Of this endeavor, a simplified method is to represent each unit by a node and the interaction between two units by an edge. Thereby the coupled complex systems are characterized by a network, and the study is correspondingly converted to investigating the control over the resultant network. Taking the distributed nature of complex networks into account, much valuable work has focused on controller design for each node of the controlled network [1–6]. However, controlling each node so that each follows a desired synchronous evolution is not always possible in practice, which can be partly attributed to the enormous quantity of nodes therein.

Many examples in reality may provide insights into the regulatory mechanisms to control networks of coupled dynamical systems. For instance, the formation of mass opinions is frequently observed in social networks; the evolution of these opinions is often driven by key individuals [7, 8]. Another example can be found in communication systems. In wireless sensor networks, cluster-head nodes are enabled to manage and coordinate the information gathered by local nodes; the deployed nodes are then organized effectively into a hierarchical network, which is of great advantage to control, computation and communication [9]. Note that controlling part of nodes such as the key individuals in social networks can also achieve the expected result. Pinning control is therefore proposed by reducing the number of controllers for synchronizing the complex networks.

The method of pinning control—apply localized feedback to a small fraction of network nodes to achieve control goal—has attracted increasing attention of researchers [10–23]. Grigoriev *et al* studied the pinning control of spatiotemporal chaos [10]. Parekh *et al* investigated the global and local control of spatiotemporal chaos in coupled map lattices [11]. Wang and Chen [13] pinned a scale-free network and investigated its synchronization stability based on a uniform complex dynamical network model. Li *et al* [14] proposed two typical pinning strategies: random pinning and selective pinning. Sorrentino *et al* [15, 16] explored the pinning controllability of complex networks in terms of the spectral properties of an extended network topology. Xiang and Chen [17] introduced Lyapunov V-stability for complex dynamical networks, and investigated pinning control based on the V-stability formulation. All these efforts offer theoretical availability of pinning control. However, we do not know how many nodes a complex network should be pinned to guarantee network synchronization. Even if the number of pinned nodes is obtained, the selection of pinned nodes from the entire network is still unknown. For example, consider a heterogeneous complex network with N nodes. There usually exist $\binom{N}{l}$ different possibilities by selecting different nodes if l nodes are expected to be pinned; yet there is great difficulty in determining which choice can ensure network synchronization since each of them may lead to a result of either achieving or losing synchronization. A recent interesting result is, if the coupling strength is large enough, the coupled dynamical network can achieve synchronization by designing only one feedback controller [21]. In particular, Zhou *et al* provided an approximation for estimating the detailed number of pinned nodes via adaptive control though the coupling matrix (Laplacian) is redefined by modifying the diagonal entries [23]. In this paper, we attempt to study the problem of pinning control by means of a linear matrix inequality (LMI) approach. We develop LMI-form criteria of pinning synchronization. Different from the existing results about pinning given in the form of an eigenvalue, the LMI approach provides a simple and clear insight into the selections of pinned nodes. Considering the reducibility of the unpinned nodes and edges between them composing the sub-network, we further investigate node selection strategies and explore their applications.

The rest of this paper is organized as follows. A uniform complex dynamical network model is introduced and some mathematical preliminaries are presented in section 2. In section 3, we investigate the linear stability of the coupled dynamical network by matrix transformation, and derive an equivalent LMI-form criterion. In section 4, the main results are given based on the LMI approach. Section 5 presents two typical selecting strategies including the analytical and numerical results of several typical network topologies. Conclusions are finally summarized in section 6.

2. Problem formulation

The complex dynamical network consisting of N identical nodes with linearly diffusive couplings under the effect of pinning control, can be formulated as follows:

$$\dot{x}_i(t) = f(x_i(t)) - \sigma \sum_{j=1}^{N} L_{ij} x_j(t) + u_i(t), \qquad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T$ is an n-dimensional vector of dynamical variables of the ith node, $f(\cdot) \in R^n$ describes the dynamics of each individual oscillator, σ is the overall coupling strength, and $L = (L_{ij}) \in R^{N\times N}$ is the Laplacian matrix, depicting the network topology. The entry $L_{ij} = L_{ji} = -1$ if there is an edge between node i and node j, otherwise $L_{ij} = 0$, and the diagonal entries of matrix L are defined by

$$L_{ii} = -\sum_{j=1, j\neq i}^{N} L_{ij}, \qquad i = 1, 2, \ldots, N.$$

The network is usually assumed to be connected without any isolated clusters, i.e., the Laplacian L is irreducible. The eigenvalues of L are real and non-negative, and can be sorted as $0 = \mu_1 < \mu_2 \leqslant \cdots \leqslant \mu_N$.

Here, the control input is generated by a simple linear feedback law:

$$u_i(t) = -\sigma k_i B_i (x_i(t) - s(t)), \qquad i = 1, 2, \ldots, N, \tag{2}$$

where B_i is a binary vector: $B_i = 1$ if node i is controlled, otherwise $B_i = 0$; the feedback gain $k_i > 0$ in order to guarantee the negativeness. The control objective is to achieve complete synchronization such that

$$x_1(t) = x_2(t) = \cdots = x_N(t) = s(t), \tag{3}$$

where synchronous state $s(t) \in R^n$ is a solution of an individual node satisfying $\dot{s}(t) = f(s)$.

Defining error vectors as $e_i(t) = x_i(t) - s(t), \forall\, 1 \leqslant i \leqslant N$, we have the error system of network (1)

$$\dot{e}_i(t) = f(x_i(t)) - f(s(t)) - \sigma \sum_{j=1}^{N} (L_{ij} + B_i k_i) e_j(t), \qquad i = 1, 2, \ldots, N. \tag{4}$$

The network (1) is admissible in terms of synchronization if the error vectors $e_i(t)$ approach zero. Suppose that l nodes in the complex network are pinned, where $l = \lfloor \delta N \rfloor$ is the integer part of the real number δN. We then need to determine the value of l and the selecting strategy.

If not otherwise specified, $L > 0$ (or $\geqslant, <, \leqslant$) means L to be a positive (or semi-positive, negative, semi-negative) definite matrix. Throughout the paper, we have the following assumption and lemmas.

Assumption 1. *Suppose that $||D_f(s)||$ is bounded, where $D_f(s)$ is the Jacobian of f evaluated at $s(t)$. That is, there exists a non-negative constant α such that $||D_f(s)|| \leqslant \alpha$.*

Lemma 1 (Schur complements [24]). *The three inequalities below are equivalent*

(1) The LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0;$$

(2) $Q(x) > 0$, $M(x) = R(x) - S^T(x) Q^{-1}(x) S(x) > 0$;
(3) $R(x) > 0$, $N(x) = Q(x) - S(x) R^{-1}(x) S^T(x) > 0$;

where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, $S(x)$ depend affinely on x.

Lemma 2 (see Taussky [25]). *An irreducible and weakly diagonally dominant matrix is nonsingular.*

Note that matrix $A = (A_{ij}) \in R^{n\times n}$ is irreducible if and only if there exists a positive integer $m \leqslant n-1$ such that $(I+A)^m > 0$, where I is an identity matrix with proper dimension; if $A_{ii} \geqslant \sum_{j=1,j\neq i}^{n} A_{ij}, \forall\, i = 1, 2, \ldots, n$, and there at least exists a certain integer $m \leqslant n$ such that $A_{mm} > \sum_{j=1,j\neq m}^{n} A_{mj}$, then A is a weakly diagonally dominant matrix.

3. Linear stability analysis

This section investigates the local pinning synchronization of a controlled dynamical network. Without loss of generality, we assume that the first l nodes $1 \leqslant l \leqslant N$ are selected and pinned under the feedback controllers $u_i(t)$.

Linearizing the controlled network (4) on the synchronous solution $s(t)$ leads to

$$\dot{\eta}_i(t) = \mathrm{D}_f(s)\eta_i(t) - \sigma \sum_{j=1}^{N} \widetilde{L}_{ij}\eta_j(t), \qquad i = 1, 2, \ldots, N, \tag{5}$$

where η_i is the vector of perturbations of the ith node, D_f is the Jacobian matrix of f on $s(t)$, and the coupling matrix is $\widetilde{L} = L + K$ with diagonal matrix $K = \mathrm{diag}(k_1, \ldots, k_l, 0, \ldots, 0)$.

Defining $\eta(t) = (\eta_1(t), \eta_2(t), \ldots, \eta_N(t)) \in R^{n\times N}$, we write equation (5) as

$$\dot{\eta}(t) = \mathrm{D}_f(s)\eta(t) - \sigma\eta(t)\widetilde{L}^T. \tag{6}$$

Since $\widetilde{L}$ is symmetric, there exists an invertible matrix $\Phi = (\phi_1, \ldots, \phi_N)$ satisfying

$$\widetilde{L}\Phi = \Phi\Lambda,$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$, and $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N$ are the eigenvalues of $\widetilde{L}$. Note that the Laplacian L is a symmetric, irreducible, and weakly diagonally dominant matrix with non-negative diagonal, we can easily derive $\widetilde{L} > 0$ according to lemma 2.

Let $\eta(t) = \xi(t)\Phi^{-1}$. From equation (6), the matrix vector $\xi(t) = (\xi_1(t), \xi_2(t), \ldots, \xi_N(t)) \in R^{n\times N}$ satisfies the following equation:

$$\dot{\xi}(t) = \mathrm{D}_f(s)\xi(t) - \sigma\xi(t)\Lambda$$

furthermore,

$$\dot{\xi}_i(t) = (\mathrm{D}_f(s) - \sigma\lambda_i I)\xi_i(t). \tag{7}$$

Thus, we have changed the linear stability problem of the synchronous evolution $s(t)$ into the stability problem of N independent n-dimensional linear systems (7). We then focus mainly on the discussion of the stability of linear systems (7).

We select $V_i = \frac{1}{2}\xi_i^T\xi_i$ as the Lyapunov function of system (7), then the derivative of V_i is given by

$$\dot{V}_i = \xi_i^T \left(\frac{D_f^T + D_f}{2} - \sigma\lambda_i I \right) \xi_i, \qquad \forall\, i = 1, 2, \ldots, N.$$

Recalling assumption 1, the above equation can be further written as

$$\dot{V}_i \leqslant (\alpha - \sigma\lambda_i)\xi_i^T\xi_i \leqslant (\alpha - \sigma\lambda_1)\xi_i^T\xi_i, \qquad \forall\, i = 1, 2, \ldots, N.$$

If $\lambda_1 > \alpha/\sigma$, $V_i \leqslant 0$. It is apparent that $\dot{V} = 0$ if and only if $e(t) = 0$. According to the Lyapunov stability theorem, the dynamical system (7) is asymptotically stable. Thus, we have the following theorem.

Theorem 1. *There must exist a positive constant λ_c such that if $\lambda_1 > \lambda_c$, the controlled complex dynamical network (1) is locally asymptotically stable about the synchronous solution $s(t)$.*

To obtain the value of λ_c is very important in our subsequent discussions. In above, a theoretical value of λ_c has been provided as α/σ. It is worth noting that such a result is just a sufficient condition of network synchronization. Besides, we can only estimate the boundary σ of very few chaotic systems such as the Lorenz, Chen, Lü systems. Therefore, it is difficult to obtain the value of α/σ in theory. A numerical value can be obtained by the largest Lyapunov exponent of equation(7). Letting $\theta = \sigma\lambda_i$ be the normalized coupling parameter, system (7) can be written as

$$\dot{w}(t) = (\mathrm{D}_f(s) - \theta I)w(t). \tag{8}$$

The largest Lyapunov exponent from equation (8) is the master stability function $\Gamma(\theta)$. If $\Gamma(\theta) < 0$, the synchronized state is stable. By calculating the value of $\Gamma(\theta)$, we can find an unbounded interval (θ_c, ∞) such that $\Gamma(\theta) < 0$, where θ_c is a positive constant. We therefore select $\lambda_c = \theta_c/\sigma$ as the critical value.

4. Main results based on LMI

Though the result in theorem 1 is quite clear, it cannot tell us how to select the pinned nodes. This section will present pinning synchronization criteria based on LMI. First, an equivalent condition of theorem 1 is given.

Theorem 2. *The inequality $\lambda_1 > \lambda_c$ holds if and only if*

$$\widetilde{L} - \lambda_c I = L + K - \lambda_c I > 0, \tag{9}$$

where λ_1 is the smallest eigenvalue of matrix $\widetilde{L}$.

Proof. A simple proof is given here. For symmetric matrix $\widetilde{L}$, there exists an orthogonal matrix U such that $U^T\widetilde{L}U = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$. Then we have

$$\widetilde{L} - \lambda_c I = U(\Lambda - \lambda_c I)U^T.$$

It is well known that $\widetilde{L} - \lambda_c I > 0$ iff $\Lambda - \lambda_c I > 0$, which is further equivalent to $\lambda_i - \lambda_c > 0, \forall i = 1, 2, \ldots, N$. Concerning the order of λ_i, we derive $\lambda_1 - \lambda_c > 0$. The proof is thus completed. □

Letting $k_1 = k_2 = \ldots = k_l = k$, the feedback gain matrix K is then written as

$$K = k\begin{bmatrix} I_l & 0 \\ 0 & 0 \end{bmatrix}, \tag{10}$$

where I_l is an $l \times l$ identity matrix. Also, the Laplacian matrix L is divided into the corresponding block form of K:

$$L = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix}, \tag{11}$$

where $L_1 \in R^{l\times l}$, $L_3 \in R^{(N-l)\times(N-l)}$ are symmetric matrices, $L_2 \in R^{l\times(N-l)}$. We have the following result.

Theorem 3. *If the value of k is sufficiently large and $\lambda_3^m > \lambda_c$, then the synchronous solutions $s(t)$ of the controlled network (1) are asymptotically stable under the pinning control laws, where λ_3^m is the smallest eigenvalue of L_3.*

Proof. Let λ_3^m and λ_3^M be the smallest and largest eigenvalues of L_3, respectively. Recalling Rayleigh–Ritz theorem [26], we have

$$\lambda_c I < \lambda_3^m I \leqslant L_3 \leqslant \lambda_3^M I. \tag{12}$$

Since the Laplacian L is a semi-positive definite matrix, we derive $PLP^T \geqslant 0$. Here, we select $P \in R^{N\times N}$ as

$$P = \begin{bmatrix} I & -L_2L_3^{-1} \\ 0 & I \end{bmatrix},$$

where L_3 is a nonsingular square matrix. Then

$$PLP^T = \begin{bmatrix} L_1 - L_2L_3^{-1}L_2^T & 0 \\ 0 & L_3 \end{bmatrix} \geqslant 0. \tag{13}$$

According to lemma 1, we deduce the following inequality from LMI (13)

$$L_1 - L_2L_3^{-1}L_2^T \geqslant 0. \tag{14}$$

Substituting LMI (12) into LMI (14) yields

$$L_1 \geqslant \frac{1}{\lambda_3^M} L_2L_2^T. \tag{15}$$

On the other hand, $\widetilde{L} - \lambda_c I$ can be written in a block form as

$$\widetilde{L} - \lambda_c I = \begin{bmatrix} L_1 + (k-\lambda_c)I & L_2 \\ L_2^T & L_3 - \lambda_c I \end{bmatrix} > 0. \tag{16}$$

Recalling lemma 1 again, we derive an equivalent condition of LMI (16) as follows:

$$L_3 - \lambda_c I > 0, \qquad L_1 + (k-\lambda_c)I - L_2(L_3 - \lambda_c I)^{-1}L_2^T > 0. \tag{17}$$

The first inequality in equation (17) is obviously true. Here we discuss the second inequality in equation (17).

Letting $L_1^* = -L_1 + L_2(L_3 - \lambda_c I_{N-l})^{-1}L_2^T$, we substitute LMI (12) and (15) into L_1^*,

$$\left(\lambda_3^m - \lambda_c\right)L_1^* \leqslant \lambda_3^M L_1. \tag{18}$$

Therefore, the inequality $\left(\lambda_3^m - \lambda_c\right)(k - \lambda_c)I > \lambda_3^M L_1$ can guarantee the positive definiteness of matrix $\widetilde{L} - \lambda_c I_N$. We thus select

$$k > \lambda_c + \frac{\lambda_3^M \lambda_1^M}{\lambda_3^m - \lambda_c}, \tag{19}$$

where λ_1^M is the largest eigenvalue of matrix L_1. Following this, the controlled complex network (1) achieves local asymptotical stability about synchronous solution $s(t)$. The proof is thus completed. □

It is worth noting that theorem 3 can estimate the approximation of feedback gain k. As we know, the smallest eigenvalue λ_1 will increase when enhancing the feedback gain k. But there is an upper bound for the function $\lambda_1(k)$ with fixed l. Thus, it is sometimes unnecessary to choose a sufficiently large value of k.

5. Two pinning strategies

The complex dynamical network (1) achieves local synchronization if $\nu_2 > \nu_c$ (without the input term, the detailed proof is omitted here since it is similar to theorem 1), where ν_2 is the second smallest eigenvalue of the Laplacian, and ν_c is a constant. As we know, $\nu_2 \leqslant d_{\min}$ for any non-complete graphs, where $d_{\min}$ is the minimum node degree [27]. We then derive that

$$d_{\min} > \nu_c, \tag{20}$$

when synchronization of a non-complete network is reached. Actually, if the synchronous evolution is given by an extra virtual node added to the original network, the controlled network (1) can be considered as a network of $N + 1$ dynamical nodes $y_i(t)$ [16], where $y_i(t) = x_i(t)$ for $i = 1, 2, \ldots, N$ and $y_{N+1} = s(t)$. We then obtain a general Laplacian matrix $\mathcal{L}$ written as

$$\mathcal{L} = \begin{bmatrix} \widetilde{L} & -B \\ 0 & 0 \end{bmatrix},$$

where $B = (k_1B_1, k_2B_2, \ldots, k_NB_N)^T$. It is easy to verify that zero and λ_i are the eigenvalues of matrix $\mathcal{L}$. Thus, the controlled complex dynamical network (1) synchronizes with $s(t)$ if the second smallest eigenvalue of matrix $\mathcal{L}$ such that $\lambda_1 > \lambda_c$, which agrees with synchronization in complex networks. Similar to the condition in equation (20), we make the following assumption:

Assumption 2. *Assume that $d_{\min} > \lambda_c$ for the controlled network (1), where $d_{\min}$ is the minimum node degree of the whole network.*

In fact, the inequality in assumption 2 holds for most cases. Suppose that there exists an unpinned node with minimum degree in network (1). If the inequality in assumption 2 does not hold, then there must exist at least a negative diagonal entry for matrix $L_3 - \lambda_c I$. It is obvious that all diagonal entries must be positive if the matrix is positive definite. In this regard, $L_3 - \lambda_c I$ cannot be a positive definite matrix, and the controlled network cannot be synchronized by such a pinning strategy. For example, most nodes have 'low' degree in scale-free networks; we have to control each 'low' node to ensure the positive definiteness of $L_3 - \lambda_c I$. That is to say, most nodes should be controlled. However the aim of pinning control is to apply localized feedback to a 'small' fraction of network nodes to guarantee synchronization. Hence, we say assumption 2 is reasonable.

For the controlled network (1), we denote by $\mathcal{C}$ the set of pinned nodes, and $\overline{\mathcal{C}}$ the set of unpinned ones. Consider a special network G' completely connected by the nodes in $\overline{\mathcal{C}}$ and edges between $\mathcal{C}$ and $\overline{\mathcal{C}}$, then

$$d_i \geqslant n_i, \tag{21}$$

where d_i is the degree of node i, n_i is the number of neighbors of the ith node. The equality occurs in equation (21) if and only if all neighbors of unpinned node i are uncontrolled nodes. In the following, we consider two cases: (1) L_3 is an irreducible matrix (i.e., all unpinned nodes are connected without any isolated clusters), and (2) L_3 is reducible (i.e., there exist several or many isolated clusters owing to deleting all pinned nodes).

5.1. L_3 is an irreducible matrix

For graph G', there must exist a certain node j such that $d_j - n_j > 0$ (or else $d_i = n_i$ for all $i \in \overline{\mathcal{C}}$. As a result, $L = G'$ or L contains at least two isolated clusters, which contradicts assumption 1). Then L_3 is an irreducible and weakly diagonally dominant matrix. According to lemma 2, we derive $L_3 > 0$. Note that the result cannot ensure network synchronization since the eigenvalue criterion does not hold. A sufficient condition can be given based on Gerschgorin's circle theorem as

$$d_i > n_i + \lambda_c, \qquad \forall i = l+1, \ldots, N. \tag{22}$$

We then perform the selecting strategies of L_3 as follows: take $\lambda_c < 1$ for example, first select a node i, then for any one connected with node j, we can select one denoted as node j to $\mathcal{C}$ and the others to the unpinned set $\overline{\mathcal{C}}$. Secondly, start with node j and put one node connected with j into $\mathcal{C}$ while others into $\overline{\mathcal{C}}$. Repeating the operation to all newly added nodes, we can find a partition of nodes. It is noted that d_i and n_i are both positive integers in equation (22), while λ_c can be any positive real number. Thus, we rewrite equation (22) as

$$d_i \geqslant n_i + \lceil \lambda_c \rceil, \qquad \forall i = l+1, \ldots, N, \tag{23}$$

where $\lceil \lambda_c \rceil$ is the nearest integer of λ_c towards infinity. In order to obtain a large set $\overline{\mathcal{C}}$, the inequality in (23) can be rewritten as an equality, i.e., $d_i = n_i + \lceil \lambda_c \rceil, \forall i = l+1, \ldots, N$.

Remark 1. The greedy method can be used to search nodes in set $\overline{\mathcal{C}}$ since such a method can ensure the irreducibility of L_3. The searching result will lead to nodes in $\overline{\mathcal{C}}$ with large degree. In other words, a given network with more large nodes probably requires a smaller number of pinned nodes by the strategy.

Corollary 1. *In k-regular networks* ($k \geqslant 2$)*, if every other* $2k$ *nodes are pinned and* $\lambda_c < 1$*, the uniform complex dynamical network* (4) *is locally synchronizable.*

Example 1. Consider a diffusively coupled dynamical network with the state equations

$$\dot{x}_i = \mathrm{D}_f x_i - \sigma \sum_{j=1}^{N} L_{ij} x_j, \qquad i = 1, 2, \ldots, N, \tag{24}$$

where the network topology is a nearest-neighbor coupled ring lattice with size $N = 9$ (shown in figure 1), the coupling strength $\sigma = 12$, and Jacobian matrix of Lorenz system with respect to the equilibrium point is

$$\mathrm{D}_f = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}.$$

It is easy to derive the critical eigenvalue $\lambda_c = 0.986$. It follows from corollary 1 that we need to control at least two nodes. Figure 2 shows the evolution of $\lambda_1(k)$ for different pinning strategies. We see that pinning nodes 4, 8 (strategy d) can achieve network synchronization when $k \geqslant 36.4$, while others, no matter what the value of k is, cannot reach synchronization. The strategy proposed in corollary 1 is the best one for improving the smallest eigenvalue of matrix $\widetilde{L}$ in k-regular network. It is observed that $\lim_{k\to\infty} \lambda_1(k)$ will approach a constant for a fixed l. That is to say, though we can enhance the synchronizability by increasing the value of feedback gain, it is of little use as k is already large enough. As a result, we had better selected more nodes to be pinned instead of enhancing feedback.

5.2. L_3 is a reducible matrix

In reality, many complex networks exhibit high heterogeneity of node connectivity, which typically possesses a power-law degree distribution. In these scale-free networks, hubs that connect many nodes of the network are quite popular [28]. To guarantee the irreducibility

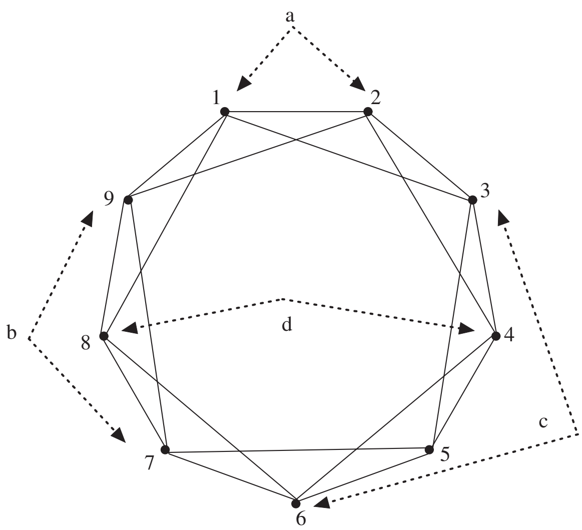

Figure 1. A two-regular network with size 9. Concerning the isomorphic feature of figure 1, there are four cases when two nodes are pinned: a: selecting nodes 1, 2; b: selecting nodes 7, 9; c: selecting nodes 3, 6; d: selecting nodes 4, 8.

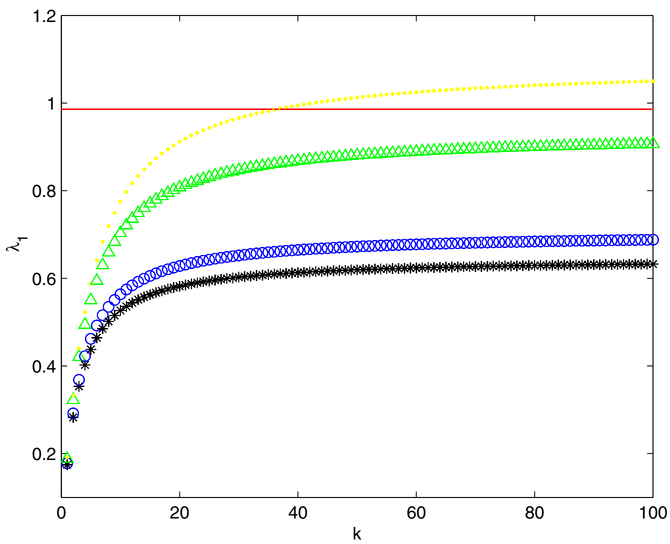

Figure 2. Evolution of $\lambda_1(k)$ under the four pinning selections as shown in figure 1. The legend is as follows: case a (stars), case b (circles), case c (triangles), case d (dots), and the critical eigenvalue of synchronization (line). If $\lambda_1(k) > \lambda_c$, then the controlled network is said to achieve local synchronization. It is easy to see that pinning the given two nodes will ensure network synchronization.

of matrix L_3 may not be suitable for a general complex network. One simple operation is to divide L_3 into several even lots of isolated clusters, i.e., L_3 is reducible.

A generic expression of L_3 is

$$L_3 = \begin{bmatrix} \Gamma & 0 & 0 & 0 \\ 0 & J_1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_o \end{bmatrix}, \tag{25}$$

where Γ is a diagonal matrix whose nonzero entries are equal to the degrees of the corresponding nodes, $J_i, i = 1, 2, \ldots, o$, are the ith isolated clusters respectively. Then we can write down the synchronization condition as

$$\Gamma > \lambda_c I \tag{26}$$

and

$$J_i > \lambda_c I, \qquad i = 1, 2, \ldots, o. \tag{27}$$

It follows from assumption 2 that equation (26) holds true for any isolated nodes. And each block J_i in equation (27) is an irreducible and weakly diagonally dominant matrix. Thus we can handle equation (27) the same as section 5.1. When there is only one isolated cluster in $\overline{\mathcal{C}}$, the case discussed above will reduce to that in section 5.1.

To obtain lots of isolated clusters, a natural idea is to select $\overline{l} = N - l$ nodes without any link between each other. Then L_3 becomes a diagonal matrix. Obviously, the larger the number $\overline{l}$, the smaller the fraction δ of pined nodes. Generally speaking, loading nodes with low degree into set $\overline{\mathcal{C}}$ will probably lead to a large dimension of matrix L_3. Such a selection is particularly effective for those networks whose topologies exhibit star-shaped architectures.

Corollary 2. *Consider a simple m-star network. If* $\lambda_c < 1$, *only pinning the kernel node will guarantee network synchronization.*

The result is clear since L_3, formed by all non-kernel nodes, is an identity matrix.

Example 2. The controlled network is described in the same way as equation (24) and the critical eigenvalue of synchronization is $\lambda_c = 0.986$. The Laplacian matrix of the star network is

$$L = \begin{bmatrix} N-1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Figure 3 shows the numerical comparison between pinning the kernel node and pinning an arbitrary non-kernel node, where network size is $N = 9$. From figure 3, we can see that the controlled network (24) with star topology will definitely synchronize to the equilibrium point by pinning the kernel node, while even the feedback gain approaches to infinity, the controlled network cannot achieve synchronization by pinning any non-kernel nodes.

Remark 2. It is not easy to obtain a diagonal matrix L_3 with the largest dimension in a general topological network. One key reason is that such a problem is the maximum independent set problem in graph, which is proved to be a commonly known NP-complete problem. In addition, it may not be the optimal solution of the number of controlled nodes.

Here we give a general theorem on how to find the set $\mathcal{C}$ and $\overline{\mathcal{C}}$.

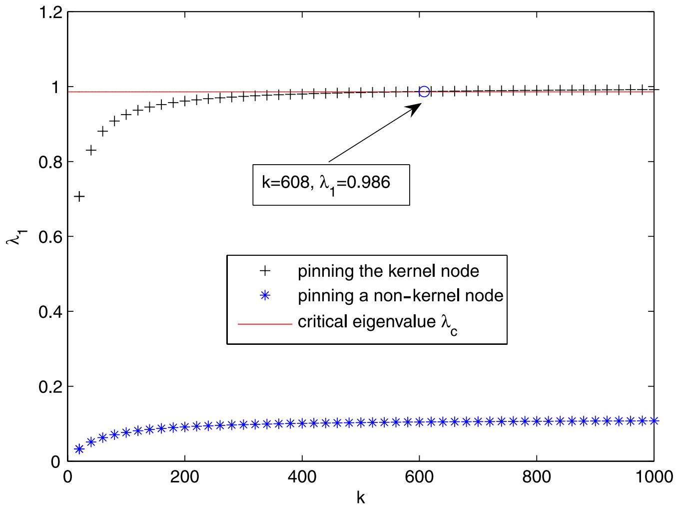

Figure 3. Pinning a star topological network. The figure shows that, if $\lambda_c = 0.986$, no matter what the k value is, the controlled network by pinning any one non-kernel node will not achieve synchronization, while the feedback gain satisfies $k > 608$, pinning the kernel node will ensure synchronization of the complex dynamical network.

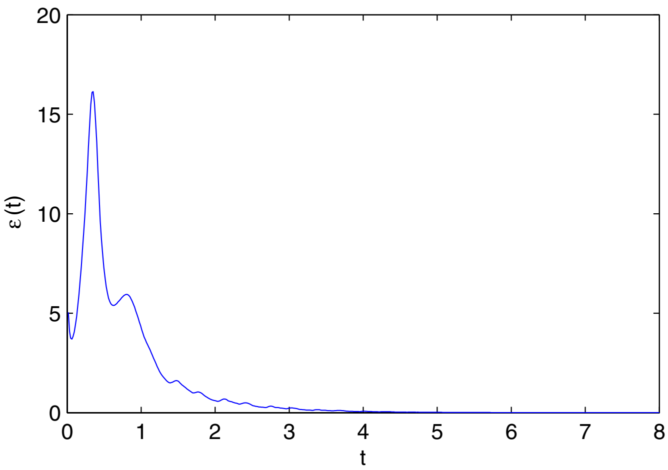

Figure 4. Pinning a BA scale-free network, where network size $N = 500$, coupling strength $\sigma = 3$, feedback gain $k = 20$, and synchronization index $\epsilon(t) = \sum_{i=1}^{N} ||e_i(t)||$, which is used to characterize network synchronization. For the network, we pinned 30 nodes to ensure the two sets in theorem 4.

Theorem 4. *Suppose that nodes in the controlled network (4) can be divided into two parts: $\mathcal{C}$ and $\overline{\mathcal{C}}$. If each node in set $\overline{\mathcal{C}}$ can be found z edges connected with nodes in set $\mathcal{C}$ and $\lambda_c < z$, then the dynamical network will synchronize with the evolution $s(t)$.*

The proof is omitted here since it can be easily deduced by the above discussions.

Example 3. In BA network models (at each time step, we add a new node with two edges that link the new node to two different nodes already present in the system) [29], any node with degree 2 cannot connect with each other. Recalling the exact degree distribution of the BA network $P(d) = \frac{12}{d(d+1)(d+2)}$ [30], we then have $P(2) = 0.5$. That is to say, almost half of the nodes inside the BA model need not to be pinned (Actually, if $\lambda_3 < 1$, similar result that nodes with degree 3 need not to be controlled can be obtained based on LMI.). On the other hand, there exist a few hubs (nodes with large degree) linking most nodes of the BA model. Then pinning these hubs will probably meet the condition of theorem 4 easily. Figure 4 supplies the numerical simulation of controlling a BA network model, where the network size $N = 500$, and the synchronous solution $s(t)$ is a chaotic Lorenz oscillator. To satisfy the condition in theorem 4, we need to select $\delta = 0.06$ at least, which is averaged over ten realizations.

Remark 3. In many evolving complex network models, such as the BA model, pseudofractual networks [31–36], and Apollonian networks [40–45] etc, nodes with the minimal degree of the corresponding networks cannot connect each other according to the growing laws. On this occasion, the diagonal matrix Γ defined in equation (25) consists of all these nodes. In other words, all these low nodes need not to be pinned any more.

6. Conclusion

In this paper, we have investigated the problem of pinning control based on LMI. Several criteria in LMI form are given to guarantee network synchronization. These results provide several pinning strategies to various typical network topologies including *k*-regular networks, star-shaped networks, and scale-free networks. We also perform corresponding numerical simulations for verification. Especially, the LMI-based criterion offers an explanation that the scale-free network will achieve synchronization more easily by selective pinning than by random pinning. It is also noted that these criteria can be easily applied to practice since the analytical results are derived with respect to node degree. Actually, many other factors such as weights, betweenness, degree correlation etc affect the controllability of complex dynamical networks. This paper just concerns node degree based on LMI, which is an attempt to study the control problem of complex networks.

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